Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series[†]

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Abstract

We give a method to embed the q-series in a (p, q)-series and derive the corresponding (p, q)-extensions of the known q-identities. The (p, q)-hypergeometric series, or twin-basic hypergeometric series (different from the usual bibasic hypergeometric series), is based on the concept of twin-basic number $[n]_{p,q} = (p^n - q^n)/(p - q)$. This twin-basic number occurs in the theory of two-parameter quantum algebras and has also been introduced independently in combinatorics. The (p, q)-identities thus derived, with doubling of the number of parameters, offer more choices for manipulations; for example, results that can be obtained via the limiting process of confluence in the usual q-series framework can be obtained by simpler substitutions. The q-results are of course special cases of the (p, q)-results corresponding to choosing $p = 1$. This also provides a new look for the q-identities.

1. Introduction

For the two-parameter quantum group $GL_{p,q}(2)$ the fundamental representation is given by the T-matrix,

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1}$$

whose elements satisfy the commutation relations

$$\begin{aligned} &ab = p^{-1}ba, \quad cd = p^{-1}dc, \quad ac = q^{-1}ca, \quad bd = q^{-1}db, \\ &bc = q^{-1}pcb, \quad ad - da = (p^{-1} - q)bc, \end{aligned} \tag{2}$$

consistent with the equation

$$R(T \otimes I)(I \otimes T) = (I \otimes T)(T \otimes I)R, \tag{3}$$

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corresponding to the R-matrix

$$R = (pq)^{1/4} \begin{pmatrix} (pq)^{-1/2} & 0 & 0 & 0 \\ 0 & (p/q)^{-1/2} & 0 & 0 \\ 0 & (pq)^{-1/2} - (pq)^{1/2} & (p/q)^{1/2} & 0 \\ 0 & 0 & 0 & (pq)^{-1/2} \end{pmatrix}. \tag{4}$$

The two-parameter quantum algebra, $U_{p,q}(gl(2))$, dual to $GL_{p,q}(2)$, is generated by $\{Z, J_0, J_\pm\}$ satisfying the commutation relations

$$\begin{aligned} &[Z, J_0] = 0, \quad [Z, J_\pm] = 0, \\ &[J_0, J_\pm] = \pm J_\pm, \quad J_+J_- - pq^{-1}J_-J_+ = \frac{p^{-2J_0} - q^{2J_0}}{p^{-1} - q}. \end{aligned} \tag{5}$$

To realize this algebra (5), a (p,q)-oscillator algebra,

$$aa^\dagger - qa^\dagger a = p^{-N}, \quad [N, a] = -a, \quad [n, a^\dagger] = a^\dagger, \tag{6}$$

was introduced in [1] generalizing/unifying several forms of q-oscillator algebras well known in the earlier physics literature related to the representation theory of single-parameter quantum algebras. The algebra (6) is satisfied when

$$a^\dagger a = \frac{p^{-N} - q^N}{p^{-1} - q}, \qquad aa^\dagger = \frac{p^{-(N+1)} - q^{N+1}}{p^{-1} - q}. \tag{7}$$

When $p = q$ or $p = 1$ the algebra (6) becomes two different versions of the q-oscillator algebra related to the representation theory of $U_q(sl(2))$.

The relations (5 and (7) suggest immediately a generalization of the Heine q-number,

$$[n]_q = \frac{1 - q^n}{1 - q}, \tag{8}$$

to a (p,q)-number as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}. \tag{9}$$

If we define a (p,q)-derivative by

$$\hat{D}_{p,q} f(z) = \frac{f(pz) - f(qz)}{(p - q)z} \tag{10}$$

then

$$\hat{D}_{p,q} z^n = [n]_{p,q} z^{n-1}. \tag{11}$$

Several properties of this (p,q)-number (9), which we will now call as the twin-basic number, including the elements of (p,q)-calculus following from (10) were studied very briefly in [1]. For the sake of convenience, we shall denote $[n]_{p,q}$ simply as $[n]$ and omit the subscripts p, q from other expressions also whenever the values of these twin-base parameters are clear from the context.

Around the same time as [1], Brodimas, *et al.* [2] and Arik, *et al.* [3] also, independently, introduced the (p,q)-number in the physics literature, but in a very much less detailed

manner. They also introduced the (p,q)-oscillator and the (p,q)-number in the same context of realization of $U_{p,q}(gl(2))$. It is a surprising fact that around the same time, without any connection to the quantum group related mathematics/physics literature, Wachs and White [4] introduced the (p,q)-number, defined as $(p^n - q^n)/(p-q)$, in the mathematics literature while generalizing the Sterling numbers, motivated by certain combinatorial problems (for further generalizations and applications in this direction see [5]). In physics literature, Katriel and Kibler [6] defined the (p,q)-binomial coefficients and derived a (p,q)-binomial theorem while discussing normal ordering for deformed boson operators obeying the algebra (6). Smirnov and Wehrhahn [7] gave an operator, or noncommutative, version of such a (p,q)-binomial theorem. Floreanini, Lapointe and Vinet [8] related the algebra (6) to bibasic hypergeometric functions [9, 10]. Burban and Klimyk [11] studied the (p,q)-differentiation, (p,q)-integration, and the (p,q)-hypergeometric series ${}_r\Psi_{r-1}$ in detail. Gelfand, *et al.* [12, 13] generalized the two-parameter deformed derivative (10) and developed a very general theory of deformation of classical hypergeometric functions. Their general formalism of deformed hypergeometric functions is based on a u-derivative

$$\hat{D}_u f(z) = \frac{1}{z} u\left(z\frac{d}{dz}\right) f(z) \tag{12}$$

where $u(z)$ is an arbitrary entire function. This leads to a u-calculus and a unified exposition of the classical theory and the q-theory and results in new u-analogues of classical hypergeometric functions. The (p,q)-hypergeometric series corresponds to the choice $u(z) = (p^z - q^z)/(p-q)$. Generalizing the definition of ${}_r\Psi_{r-1}$ by Burban and Klimyk [11], one of us defined the general (p,q)-hypergeometric series ${}_r\Phi_s$ and derived some related preliminary results [14]. Some applications of the (p,q)-hypergeometric series in the context of representations of two-parameter quantum groups have been considered by Nishizawa [15] and Sahai and Srivastava [16].

In the present work we shall deal only with the (p,q)-hypergeometric series as defined in [14]. We introduce a method of application of the (p,q)-series to convert the various well known q-identities into their (p,q)-analogues; after the conversion the resulting (p,q)-identities offer more choices for symbolic manipulations transcending the applications of the original q-identities and in fact give a new look to the latter.

2. Twin-basic hypergeometric series ${}_r\Phi_s$

Let us recall some basic definitions from the theory of q-hypergeometric series [17]. The q-shifted factorial is given by

$$(a;q)_n \quad = \quad \begin{cases} 1, \quad n=0, \\ (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1}), \\ \qquad n = 1,2,\ldots. \end{cases} \tag{13}$$

With

$$(a_1, a_2, \ldots, a_k; q)_n = (a_1;q)_n (a_2;q)_n \ldots (a_k;q)_n. \tag{14}$$

the q-hypergeometric series, or the basic hypergeometric series, ${}_r\phi_s$ is defined as

$$\begin{aligned} & {}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z) \\ & \quad = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n (q;q)_n} ((-1)^n q^{n(n-1)/2})^{1+s-r} z^n. \end{aligned} \tag{15}$$

Let us now call the (p,q)-number (9) as twin-basic number and define the twin-basic analogues of (13) and (14) as follows:

$$((a,b);(p,q))_n \;=\; \begin{cases} 1, \qquad n=0, \\ (a-b)(ap-bq)(ap^2-bq^2)\ldots(ap^{n-1}-bq^{n-1}), \\ \qquad n=1,2,\ldots. \end{cases} \tag{16}$$

$$\begin{aligned}&((a_{1p},a_{1q}),(a_{2p},a_{2q}),\ldots,(a_{mp},a_{mq});(p,q))_n \\ &\quad = ((a_{1p},a_{1q});(p,q))_n((a_{2p},a_{2q});(p,q))_n\ldots((a_{mp},a_{mq});(p,q))_n.\end{aligned} \tag{17}$$

Note that

$$(a;q)_n = ((1,a);(1,q))_n. \tag{18}$$

Then, the (p,q)-analogue of (15), the (p,q)-hypergeometric series, or the twin-basic hypergeometric series, can be defined as

$$\begin{aligned}&{}_r\Phi_s((a_{1p},a_{1q}),\ldots,(a_{rp},a_{rq});(b_{1p},b_{1q}),\ldots,(b_{sp},b_{sq});(p,q),z) \\ &= \sum_{n=0}^{\infty}\frac{((a_{1p},a_{1q}),\ldots,(a_{rp},a_{rq});(p,q))_n}{((b_{1p},b_{1q}),\ldots,(b_{sp},b_{sq});(p,q))_n((p,q);(p,q))_n} \\ &\qquad\qquad\qquad\qquad \times((-1)^n(q/p)^{n(n-1)/2})^{1+s-r}z^n,\end{aligned} \tag{19}$$

with $|q/p|<1$ [14]. Though, generally, we shall assume $0<q<p$, p and q can also take other values if there is no problem with convergence of the particular series involved in a result. When $a_{1p}=a_{2p}=\ldots=a_{rp}=b_{1p}=b_{2p}=\ldots=b_{sp}=1$, $a_{1q}=a_1, a_{2q}=a_2,\ldots,a_{rq}=a_r$, $b_{1q}=b_1,b_{2q}=b_2,\ldots,b_{s,q}=b_s$, and $p=1$, ${}_r\Phi_s \longrightarrow {}_r\phi_s$. Special interesting choices for (p,q), from the point of view of quantum groups, are $(q^{-1/2},q^{1/2})$, (q^{-1},q) and, more generally, (p^{-1},q). Throughout the paper we shall assume $|z|<1$. Also, we shall assume all the parameters to be generic, with nonzero values, unless specified otherwise. While referring to the classical results of the q-series we shall use the standard notations as in [17] (see also [21]). Often, the parameter doublets (a_p,a_q), (b_p,b_q), etc., will be denoted by different symbols according to the convenience of the situation and such notations should be clear from the context.

Let us recall the definition of a bibasic hypergeometric series with two bases q and q_1 [9, 10] (see also [17]):

$$\begin{aligned}&\mathcal{F}(\underline{a},\underline{c};\underline{b},\underline{d};q,q_1,z) = \\ &\quad \sum_{n=0}^{\infty}\frac{(\underline{a};q)_n(\underline{c};q_1)_n}{(\underline{b};q)_n(\underline{d};q_1)_n(q;q)_n} \\ &\quad \times((-1)^nq^{n(n-1)/2})^{1+s-r}((-1)^nq_1^{n(n-1)/2})^{s_1-r_1}z^n,\end{aligned} \tag{20}$$

where $\underline{a}=(a_1,a_2,\ldots,a_r)$, $\underline{b}=(b_1,b_2,\ldots,b_s)$, $\underline{c}=(c_1,c_2,\ldots,c_{r_1})$, and $\underline{d}=(d_1,d_2,\ldots,d_{s_1})$. It is clear that in (20) the two unconnected bases q and q_1 are regarded are assigned partially to different numerator and denominator parameters whereas in the twin-basic hypergeometric series (19) the twin base parameters p and q are inseparable and assigned to all the numerator and denominator parameter doublets.

Let

$$\Delta_{(\alpha,\beta)}f(z) = \alpha f(qz)-\beta f(pz). \tag{21}$$

With

$$\Delta f(z) = \Delta_{(1,1)} f(z) = f(qz) - f(pz), \tag{22}$$

it may be noted that

$$\hat{D} f(z) = \frac{\Delta f(z)}{\Delta z}. \tag{23}$$

Then it is seen that ${}_r\Phi_s$ satisfies the (p,q)-difference equation

$$\left(\Delta \prod_{i=1}^{s} \Delta_{(b_{iq}/q, b_{ip}/p)} \right) {}_r\Phi_s = \left(z \prod_{i=1}^{r} \Delta_{(a_{iq}, a_{ip})} \right) {}_r\Phi_s \left((q/p)^{1+s-r} z \right). \tag{24}$$

When $a_{1p} = a_{2p} = \ldots = a_{rp} = b_{1p} = b_{2p} = \ldots = b_{sp} = 1$, $a_{1q} = a_1, a_{2q} = a_2, \ldots, a_{rq} = a_r$, $b_{1q} = b_1, b_{2q} = b_2, \ldots, b_{s,q} = b_s$, and $p = 1$ this equation reduces to the q-difference equation satisfied by ${}_r\phi_s$.

Let us now construct a method to embed the usual ${}_r\phi_s$-series (15) in the ${}_r\Phi_s$-series (19). To this end, we note

$$((la, lb); (p,q))_n = l^n ((a,b); (p,q))_n, \tag{25}$$

for any arbitrary nonzero l, and

$$(b/a; q/p)_n = a^{-n} p^{-n(n-1)/2} ((a,b); (p,q))_n. \tag{26}$$

Thus, we can write, formally,

$$\begin{aligned}
&{}_r\phi_s(a_{1q}/a_{1p}, a_{2q}/a_{2p}, .., a_{rq}/a_{rp}; b_{1q}/b_{1p}, b_{2q}/b_{2p}, .., b_{sq}/b_{sp}; q/p, z) \\
&= \begin{cases}
{}_r\Phi_s((a_{1p}, a_{1q}), .., (a_{rp}, a_{rq}); (b_{1p}, b_{1q}), .., (b_{sp}, b_{sq}); (p,q), \mu z) \\
\qquad \text{if } s = r-1, \\
{}_{s+1}\Phi_s((a_{1p}, a_{1q}), .., (a_{rp}, a_{rq}), (0,1), .., (0,1); (b_{1p}, b_{1q}), .., (b_{sp}, b_{sq}); \\
\qquad (p,q), \mu z), \quad \text{if } s > r-1, \\
{}_r\Phi_{r-1}((a_{1p}, a_{1q}), .., (a_{rp}, a_{rq}); (b_{1p}, b_{1q}), .., (b_{sp}, b_{sq}), (0,1), .., (0,1); \\
\qquad (p,q), \mu z), \quad \text{if } s < r-1,
\end{cases} \\
&\qquad \text{with } \mu = \frac{b_{1p} b_{2p} .. b_{sp} p}{a_{1p} a_{2p} .. a_{rp}},
\end{aligned} \tag{27}$$

assuming that the given ${}_r\phi_s$-series is convergent or terminating. Hence any well behaved ϕ-series can be written as a Φ-series. But, the converse is not true, in general; in the general case, when $p \neq 1$, this is possible only for an ${}_r\Phi_{r-1}$. To see this, it is enough to look at ${}_0\Phi_0$:

$$\begin{aligned}
{}_0\Phi_0(-;-;(p,q),z) &= \sum_{n=0}^{\infty} \frac{(-1)^n (q/p)^{n(n-1)/2}}{((p,q);(p,q))_n} z^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (\rho/p)^{n(n-1)/2}}{(\rho;\rho)_n} (z/p)^n, \quad \text{with } \rho = q/p,
\end{aligned} \tag{28}$$

which shows that ${}_0\Phi_0$ becomes a ϕ-series if and only if $p = 1$. Similarly, one is easily convinced that a generic ${}_r\Phi_s$-series cannot be identified within the class of ϕ-series unless $p = 1$ or $s = r-1$ (the first case in the above equation (27)). It is thus clear that the (p,q)-series is a larger structure in which the q-series gets embedded. Also, note that in

the usual theory of ϕ-series there is no direct analogue for the choice $a_{ip} = 0$ or $b_{ip} = 0$, for any i, permissible, in general (of course, subject to conditions of convergence and so on), in the case of the (p, q)-series; to obtain a corresponding result in the case of the ϕ-series one will have to resort to the limit process of confluence, namely, replacing z by z/a_r and taking the limit $a_r \longrightarrow \infty$. As an example consider the following. As is well known, in the definition of the usual q-hypergeometric series (15), presence of the factor $((-1)^n q^{n(n-1)/2})^{1+s-r}$ (absent in the earlier literature [18, 19, 20]) leads to the useful relation

$$\lim_{a_r \longrightarrow \infty} {}_r\phi_s(z/a_r) = {}_{r-1}\phi_s(z). \tag{29}$$

For the (p, q)-hypergeometric series (19) the corresponding property is:

$$\begin{aligned}
\lim_{a_{rq} \longrightarrow \infty} {}_r\Phi_s(z/a_{rq}) &= {}_r\Phi_s((a_{1p}, a_{1q}), .., (a_{(r-1)p}, a_{(r-1)q})(0, 1);\\
&\qquad (b_{1p}, b_{1q}), .., (b_{sp}, b_{sq}); (p, q), z)\\
\lim_{a_{rp} \longrightarrow \infty} {}_r\Phi_s(z/a_{rp}) &= {}_r\Phi_s((a_{1p}, a_{1q}), .., (a_{(r-1)p}, a_{(r-1)q})(1, 0);\\
&\qquad (b_{1p}, b_{1q}), .., (b_{sp}, b_{sq}); (p, q), z).
\end{aligned} \tag{30}$$

Let us also note down the converse of (27) in the case $s = r - 1$:

$$\begin{aligned}
&{}_r\Phi_{r-1}((a_{1p}, a_{1q}), .., (a_{rp}, a_{rq}); (b_{1p}, b_{1q}), .., (b_{r-1,p}, b_{r-1,q}); (p, q), z)\\
&= {}_r\phi_{r-1}(a_{1q}/a_{1p}, a_{2q}/a_{2p}, .., a_{rq}/a_{rp};\\
&\qquad b_{1q}/b_{1p}, b_{2q}/b_{2p}, .., b_{r-1,q}/b_{r-1,p}; q/p, z/\mu).
\end{aligned} \tag{31}$$

Another set of relations often useful are

$$\begin{aligned}
\frac{(b/a; q/p)_\infty}{(d/c; q/p)_\infty} &= \lim_{N \longrightarrow \infty} \frac{(b/a; q/p)_N}{(d/c; q/p)_N}\\
&= \lim_{N \longrightarrow \infty} \frac{a^{-N} p^{-N(N-1)/2}((a, b); (p, q))_N}{c^{-N} p^{-N(N-1)/2}((c, d); (p, q))_N}\\
&= \frac{((c, bc/a); (p, q))_\infty}{((c, d); (p, q))_\infty}\\
&= \frac{((a, b); (p, q))_\infty}{((a, ad/c); (p, q))_\infty},
\end{aligned} \tag{32}$$

and its obvious generalizations containing several factors in the numerator and denominator.

Manipulations using the above relations take the usual q-identities to (p, q)-identities. The original q-identities are, of course, special cases corresponding to the choice $a_{1p} = a_{2p} = \ldots a_{rp} = b_{1p} = b_{2p} = \ldots = b_{r-1,p} = 1$, and $p = 1$. We shall consider a few examples below.

3. (p, q)-Binomial theorem

The usual q-binomial theorem is

$$ {}_1\phi_0(a; -; q, z) = \frac{(az; q)_\infty}{(z; q)_\infty}. \tag{33}$$

The (p,q)-analogue of this is given by

$$ {}_1\Phi_0((a,b);-;(p,q),z) = \frac{((p,bz);(p,q))_\infty}{((p,az);(p,q))_\infty}. \tag{34}$$

Proof: Let us rewrite (33) as

$$ {}_1\phi_0(b/a;-;q/p,\zeta) = \frac{(b\zeta/a;q/p)_\infty}{(\zeta;q/p)_\infty}. \tag{35}$$

Using (27) and (32), we have

$$ {}_1\Phi_0((a,b);-;(p,q),p\zeta/a) = \frac{((a,b\zeta);(p,q))_\infty}{((a,a\zeta);(p,q))_\infty}. \tag{36}$$

Now, taking $\zeta = za/p$, we get

$$ {}_1\Phi_0((a,b);-;(p,q),z) = \frac{((a,abz/p);(p,q))_\infty}{((a,a^2z/p);(p,q))_\infty}. \tag{37}$$

Using the arguments of (25) and (32), by pulling out powers of a/p in the numerator and denominator of the r.h.s., the (p,q)-binomial theorem (34) follows.

The usual q-binomial theorem (33) is recovered when $a=1$ and $p=1$. The (p,q)-binomial theorem obtained in [11] is a special case of (34) corresponding to the specific choice $(a,b) = (q^{-a/2}, p^{a/2})$ and $(p,q) = (q^{-1/2}, p^{1/2})$. An interesting feature of the (p,q)-binomial theorem (34) may be noted here. The product $\prod_{i=1}^{n} {}_1\Phi_0((a_{ip},a_{iq});-;(p,q),z)$ is seen to be an invariant under the group of independent permutations of the p-components $(a_{1p},a_{2p},\ldots,a_{np})$ and the q-components $(a_{1q},a_{2q},\ldots,a_{nq})$. This product has value 1 if the n-tuple of p-components $(a_{1p},a_{2p},\ldots,a_{np})$ is related to the n-tuple of q-components $(a_{1q},a_{2q},\ldots,a_{nq})$ by a mere permutation.

For $n=2$ this result implies that

$$ {}_1\Phi_0((a,b);-;(p,q),z)\,{}_1\Phi_0((b,a);-;(p,q),z) = 1. \tag{38}$$

A special case of this relation is

$$ {}_1\Phi_0((1,0);-;(1,q),z)\,{}_1\Phi_0((0,1);-;(1,q),z) = 1. \tag{39}$$

Recognizing that

$$\begin{aligned} {}_1\Phi_0((1,0);-;(1,q),z) &= \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} z^n = e_q(z), \\ {}_1\Phi_0((0,1);-;(1,q),z) &= \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} (-z)^n = E_q(-z), \end{aligned} \tag{40}$$

where $e_q(z)$ and $E_q(z)$ are the canonical q-exponentials, the well known relation

$$ e_q(z)E_q(-z) = 1, \tag{41}$$

follows from (39). It should be noted that, while in the usual q-theory [17] $e_q(z)$ is ${}_1\phi_0(0;-;q,z)$ and $E_q(z)$ is ${}_0\phi_0(-;-;q,-z)$, in the (p,q)-series formalism both $e_q(z)$ and $E_q(z)$ belong to the same ${}_1\Phi_0$-series. This result suggests the natural definitions

$$e_{p,q}(z) = {}_1\Phi_0((1,0);-;(p,q),z) \;=\; \sum_{n=0}^{\infty} \frac{p^{n(n-1)/2}}{((p,q);(p,q))_n} z^n, \tag{42}$$

$$E_{p,q}(z) = {}_1\Phi_0((0,1);-;(p,q),-z) \;=\; \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{((p,q);(p,q))_n} z^n, \tag{43}$$

for the (p,q)-exponentials such that

$$e_{p,q}(z)E_{p,q}(-z) = 1. \tag{44}$$

For $p = 1$, $e_{1,q}(z)$ and $E_{1,q}(z)$ become $e_q(z)$ and $E_q(z)$ respectively.

For $n = 3$ the above general result and the relation (38) imply

$$\begin{aligned} &{}_1\Phi_0((u,v);-;(p,q),z){}_1\Phi_0((v,w);-;(p,q),z) \\ &\quad = {}_1\Phi_0((u,w);-;(p,q),z). \end{aligned} \tag{45}$$

Now, if we take $u = 1$, $v = a$, $w = ab$ and $p = 1$ then this equation (45) is just the well known product formula for ${}_1\phi_0$, namely,

$${}_1\phi_0(a;-;q,z){}_1\phi_0(b;-;q,az) = {}_1\phi_0(ab;-;q,z)\,, \tag{46}$$

in view of the relation (31). Thus we get a new way of looking at the product formula (46) within the (p,q)-series formalism.

4. (p,q)-Binomial coefficient

The definition

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{((p,q);(p,q))_n}{((p,q);(p,q))_k((p,q);(p,q))_{n-k}}, \quad k = 0,1,\ldots,n, \tag{47}$$

provides a natural generalization of the q-binomial coefficient. In terms of the (p,q)-number the (p,q)-binomial coefficient (written without the subscript p,q) becomes

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}, \tag{48}$$

where, as usual,

$$[n]! = [n][n-1]\ldots[2][1], \qquad [0]! = 1. \tag{49}$$

Then, the result

$$\begin{aligned} {}_1\Phi_0((p^n,q^n);-;(p,q),z) &= \sum_{k=0}^{\infty} \begin{bmatrix} n-1+k \\ k \end{bmatrix} z^k \\ = \frac{p^{n(n+1)/2}}{((p,p^n z);(p,q))_n} &= \left\{ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (pq)^{k(k-1)/2}(-z)^k \right\}^{-1}, \end{aligned} \tag{50}$$

follows by taking $a = p^n$ and $b = q^n$ in (38). The relation (50) is obviously a generalization of the result

$$\begin{aligned} {}_1\phi_0(q^n; -; q, z) &= \sum_{k=0}^{\infty} \left[\begin{array}{c} n-1+k \\ k \end{array} \right]_q z^k = \frac{1}{(z;q)_n} \\ &= \left\{ \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{k(k-1)/2} (-z)^k \right\}^{-1}. \end{aligned} \tag{51}$$

If we take $p = 0$ in (50) we get, correctly of course,

$$\sum_{k=0}^{\infty} (q^{n-1}z)^k = \frac{1}{1 - q^{n-1}z}. \tag{52}$$

It should be noted that there is no analogue for the choice $p = 0$ in the usual q-series formalism. We can also take the limit $p \longrightarrow q \neq 1$. Then, the equation (50) takes the form

$$\begin{aligned} {}_1F_0(n; -; q^{n-1}z) &= \sum_{k=0}^{\infty} \left(\begin{array}{c} n-1+k \\ k \end{array} \right) (q^{n-1}z)^k \\ &= (1 - q^{n-1}z)^{-n} = \left\{ \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) (-q^{n-1}z)^k \right\}^{-1}. \end{aligned} \tag{53}$$

Thus, it is seen that, though a (p,q)-identity may be derived starting with a q-identity, the (p,q)-identity offers more choices for manipulations. If we choose $(p,q) = (q^{-1}, q)$, then, the identity (50) becomes

$$\begin{aligned} {}_1\Phi_0((q^{-n}, q^n); -; (q^{-1}, q), z) &= \sum_{k=0}^{\infty} \left[\begin{array}{c} n-1+k \\ k \end{array} \right]_{q^{-1},q} z^k \\ &= \frac{q^{-n(n+1)/2}}{((q^{-1}, zq^{-n}); (q^{-1}, q))_n} = \left\{ \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{q^{-1},q} (-z)^k \right\}^{-1}, \end{aligned} \tag{54}$$

with

$$\left[\begin{array}{c} n \\ k \end{array} \right]_{q^{-1},q} = \frac{((q^{-1}, q); (q^{-1}, q))_n}{((q^{-1}, q); (q^{-1}, q))_k ((q^{-1}, q); (q^{-1}, q))_{n-k}}, \quad k = 0, 1, \ldots, n. \tag{55}$$

which should be relevant in the context of quantum groups.

From (50), let us take

$$p^{-n(n+1)/2}((p, p^n z); (p,q))_n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] (pq)^{k(k-1)/2} (-z)^k. \tag{56}$$

Using (25) and taking $z = \zeta_q/\zeta_p$, we can rewrite (56) as

$$((p\zeta_p, p^n\zeta_q); (p,q))_n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] p^{(n(n+1)+k(k-1))/2} q^{k(k-1)/2} (-1)^k \zeta_q^k \zeta_p^{n-k}. \tag{57}$$

Now, renaming $p\zeta_p$ and $p^n\zeta_q$ as a and b, respectively, we get

$$((a,b);(p,q))_n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] (-1)^k p^{(n-k)(n-k-1)/2} q^{k(k-1)/2} a^{n-k} b^k. \tag{58}$$

The (p,q)-binomial theorem derived in [6], using the recursion relations of the (p,q)-binomial coefficients, corresponds to (58) with the notations $a = l$, $b = -x$.

An operator, or noncommutative, form of the q-binomial theorem is known [17]: If x and y are noncommuting variables such that $xy = qyx$, q commutes with x and y, and the associative law holds, then

$$(x+y)^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q y^k x^{n-k} = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{q^{-1}} x^k y^{n-k}. \tag{59}$$

A (p,q)-extension of this result is derived in [7], in a specific context of a quantum group. This result can be stated in a general form as follows:

$$(ax+by)^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} a^{n-k} b^k y^k x^{n-k} = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p^{-1},q^{-1}} b^{n-k} a^k x^k y^{n-k}, \tag{60}$$

where $ab = p^{-1}ba$, $xy = qyx$, all other commutators among the variables $\{a,b,x,y\}$ vanish, p and q commute with $\{a,b,x,y\}$, and the associative law holds. Proof of (60) follows by replacing in (59) q by q/p and (x,y) by (ax,by), and reexpressing the result in terms of p, q, a, b, x, and y. In deriving the second part of (60) one has to use the formula

$$((a,b);(p,q))_n = (-1)^n a^n b^n (pq)^{n(n-1)/2} ((a^{-1},b^{-1});(p^{-1},q^{-1}))_n. \tag{61}$$

5. (p,q)-Heine transformation for ${}_2\Phi_1$

The Heine transformation of the ${}_2\phi_1$ series, namely,

$${}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty} {}_2\phi_1(c/b,z;az;q,b)\,, \tag{62}$$

has the following (p,q)-analogue:

$$\begin{aligned} &{}_2\Phi_1((a,b),(c,d);(e,f);(p,q),z) \\ &= \frac{((ce,de),(pe,bcz);(p,q))_\infty}{(ce,cf),(pe,acz);(p,q))_\infty} \\ &\quad \times {}_2\Phi_1((de,cf),(pe,acz);(pe,bcz);(p,q),p/ce), \end{aligned} \tag{63}$$

Proof: By the Heine transformation (62)

$$\begin{aligned} &{}_2\phi_1(b/a,d/c;f/e;q/p,\zeta) \\ &\quad = \frac{(d/c,b\zeta/a;q/p)_\infty}{(f/e,\zeta;q/p)_\infty} {}_2\phi_1(cf/de,\zeta;b\zeta/a;q/p,d/c). \end{aligned} \tag{64}$$

Using (27) and following arguments of the type used in (32) we can rewrite this equation as

$$\begin{aligned} &{}_2\Phi_1((a,b),(c,d);(e,f);(p,q),pe\zeta/ac) \\ &\quad = \frac{((c,d),(a,b\zeta);(p,q))_\infty}{((e,f),(ac/e,ac\zeta/e);(p,q))_\infty} \\ &\quad\quad {}_2\Phi_1((de,cf),(1,\zeta);(a,b\zeta);(p,q),pa/ce). \end{aligned} \tag{65}$$

Now, taking $\zeta = acz/pe$, we get

$$
\begin{aligned}
&{}_2\Phi_1((a,b),(c,d);(e,f);(p,q),z) \\
&\quad = \frac{((ce,de),(pe,bcz);(p,q))_\infty}{(ce,cf),(pe,acz);(p,q))_\infty} \\
&\quad {}_2\Phi_1((de,cf),(pe,acz);(pe,bcz);(p,q),p/ce),
\end{aligned} \tag{66}
$$

thus, arriving at the (p,q)-Heine transformation formula (63) for ${}_2\Phi_1$.

Setting $a = 0$, $b = c = e = 1$, relabeling d as a and f as b, and taking $p = 1$, in (63) we obtain the transformation

$$
{}_1\phi_1(a;b;q,z) = \frac{(a,z;q)_\infty}{(b;q)_\infty} {}_2\phi_1(0,b/a;z;q,a)\,, \tag{67}
$$

which can be directly derived from the q-Heine transformation formula (62) by using the limiting process of confluence, namely, replacing z by z/a and taking the limit $a \longrightarrow \infty$, and then relabeling the parameters. Now, taking $z = b/a$ in (67) one obtains, using the q-binomial theorem, the summation formula [17]

$$
{}_1\phi_1(a;b;q,b/a) = \frac{(b/a;q)_\infty}{(b;q)_\infty}\,, \tag{68}
$$

which can also be obtained from the (p,q)-Gauss sum (70), given below, with the same choice of parameters.

6. (p,q)-Gauss sum

Using the Heine transformation (62) one obtains the q-Gauss sum

$$
{}_2\phi_1(a,b;c;q,c/ab) = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}, \qquad |c/ab| < 1. \tag{69}
$$

The (p,q)-Gauss sum takes the form

$$
\begin{aligned}
&{}_2\Phi_1((a,b),(c,d);(e,f);(p,q),pf/bd) \\
&\quad = \frac{((be,af),(de,cf);(p,q))_\infty}{((e,f),(bde,acf);(p,q))_\infty}, \qquad |acf/bde| < 1.
\end{aligned} \tag{70}
$$

Proof: Let $z = pf/bd$ in the (p,q)-Heine transformation formula (63). The result is

$$
\begin{aligned}
&{}_2\Phi_1((a,b),(c,d);(e,f);(p,q),pf/bd) \\
&\quad = \frac{((ce,de),(pe,pcf/d);(p,q))_\infty}{((ce,cf),(pe,pacf/bd);(p,q))_\infty} \\
&\qquad \times {}_2\Phi_1((de,cf),(pe,pacf/bd);(pe,pcf/d);(p,q),p/ce), \\
&\quad = \frac{((ce,de),(bde,bcf);(p,q))_\infty}{((ce,cf),(bde,acf);(p,q))_\infty} \\
&\qquad \times {}_2\Phi_1((de,cf),(pe,pacf/bd);(pe,pcf/d);(p,q),p/ce).
\end{aligned} \tag{71}
$$

Note that

$$
\begin{aligned}
&{}_2\Phi_1((de,cf),(pe,pacf/bd);(pe,pcf/d);(p,q),p/ce)\\
&\quad = {}_2\Phi_1((de,cf),(bde,acf);(bde,bcf);(p,q),p/ce)\\
&\quad = {}_1\Phi_0((bde,acf);-;(p,q),p/bce)\\
&\quad = \frac{((p,paf/be);(p,q))_\infty}{((p,pd/c);(p,q))_\infty} = \frac{((be,af);(p,q))_\infty}{((be,bde/c);(p,q))_\infty},
\end{aligned}
\tag{72}
$$

in view of the (p,q)-binomial theorem and (32). Hence,

$$
\begin{aligned}
&{}_2\Phi_1((a,b),(c,d);(e,f);(p,q),pf/bd)\\
&\quad = \frac{((ce,de),(bde,bcf),(be,af);(p,q))_\infty}{((ce,cf),(bde,acf),(be,bde/c);(p,q))_\infty}\\
&\quad = \frac{((c,d),(de,cf),(be,af);(p,q))_\infty}{((e,f),(bde,acf),(c,d);(p,q))_\infty}\\
&\quad = \frac{((de,cf),(be,af);(p,q))_\infty}{((e,f),(bde,acf);(p,q))_\infty}.
\end{aligned}
\tag{73}
$$

Thus, the (p,q)-Gauss sum (70) is derived.

The identity

$$
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q,qz;q)_n} z^n = \frac{1}{(qz;q)_\infty},
\tag{74}
$$

is usually obtained from the q-Gauss sum (69) by setting $c = qz$ and letting $a \longrightarrow \infty$ and $b \longrightarrow \infty$. It should be noted that this identity follows immediately from the (p,q)-Gauss sum (70) by mere substitution $a = c = 0$, $b = d = e = 1$, $f = qz$ and $p = 1$.

Another useful form of (70) is

$$
\begin{aligned}
&{}_2\Phi_1((a,1),(b,c);(d,\sigma c);(p,q),\sigma p)\\
&\quad = \frac{((d,\sigma ac),(d,\sigma b);(p,q))_\infty}{((d,\sigma c),(d,\sigma ab);(p,q))_\infty}. \qquad |\sigma ab/d| < 1.
\end{aligned}
\tag{75}
$$

Now, substituting in (75) $a = c = 0$, $b = d = 1$, $\sigma = \sqrt{q}z$ and $p = 1$, one gets another well-known identity

$$
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q;q)_n} z^n = (\sqrt{q}z;q)_\infty,
\tag{76}
$$

which is usually obtained from the q-Gauss sum (69) by setting $c = \sqrt{q}bz$ and then letting $b \longrightarrow 0$ and $a \longrightarrow \infty$. These examples illustrate the usefulness of the (p,q)-series formalism even for the treatment of the usual q-series.

7. (p,q)-Ramanujan sum

Let us assume the obvious (p,q)-generalizations of the basic notations and definitions associated with bilateral q-hypergeometric series. Thus, we write

$$
((a,b);(p,q))_{-n} = \frac{1}{((ap^{-n},bq^{-n});(p,q))_n}
$$

$$= \frac{1}{(ap^{-1}-bq^{-1})(ap^{-2}-bq^{-2})\dots(ap^{-n}-bq^{-n})}$$
$$= \frac{(-pq/ab)^n (pq)^{n(n-1)/2}}{((p/a,q/b);(p,q))_n}, \tag{77}$$

and

$${}_1\Psi_1((a,b);(c,d);(p,q),z) = \sum_{n=-\infty}^{\infty} \frac{((a,b);(p,q))_n}{((c,d);(p,q))_n} z^n$$
$$= \sum_{n=0}^{\infty} \frac{((a,b);(p,q))_n}{((c,d);(p,q))_n} z^n + \sum_{n=1}^{\infty} \frac{((p/c,q/d);(p,q))_n}{((p/a,q/b);(p,q))_n} \left(\frac{cd}{abz}\right)^n. \tag{78}$$

One can show that

$${}_1\Psi_1((a,b);(c,d);(p,q),z) = {}_1\psi_1(b/a;d/c;q/p,za/c) \tag{79}$$

where ${}_1\psi_1$ is the usual bilateral q-series. Then, using the Ramanujan sum,

$${}_1\psi_1(a;b;q,z) = \frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}, \qquad |b/a| < |z| < 1, \tag{80}$$

one can show that the (p,q)-analogue of the Ramanujan sum is

$${}_1\Psi_1((a,b);(c,d);(p,q),z)$$
$$= \frac{((p,q),(bc,ad),(c,bz),(pbz,qc);(p,a))_\infty}{((c,d),(pb,qa),(c,az),(pbz,pd);(p,a))_\infty},$$
$$|ad/bc| < |z| < 1. \tag{81}$$

To obtain the (p,q)-analogue of the Jacobi triple product identity from this the steps are: (i) $(a,b) \longrightarrow (1/a,1/b)$, $z \longrightarrow zb/a$, (ii) $d = 0$, $(p,q) \longrightarrow (p^2,q^2)$, $z \longrightarrow zq/p$, (iii) $b \longrightarrow 0$, and (iv) $(p,q) \longrightarrow (\sqrt{p},\sqrt{q})$. The result is:

$$\sum_{n=-\infty}^{\infty} (-1)^n (q/p)^{n^2/2} (z/ac)^n$$
$$= \frac{((p,q),(\sqrt{p}ca,\sqrt{q}z),(\sqrt{p}z,\sqrt{q}ca);(p,q))_\infty}{((p,0),(\sqrt{p}ca,0),(\sqrt{p}z,0);(p,q))_\infty}, \tag{82}$$

which is same as the well known q-result with the replacements $q \longrightarrow q/p$ and $z \longrightarrow z/ac$. The usual Jacobi triplet product identity can also be obtained in a simpler way directly from ${}_1\Psi_1$ by letting $a = d = 0$, $b = c = 1$, $p = 1$ and $z \longrightarrow z\sqrt{q}$.

Taking $ac = 1$ in (82), we can also write the (p,q)-analogue of the Jacobi triple product, for $q < p$, $|z| < 1$, as

$$\sum_{n=-\infty}^{\infty} (-1)^n (q/p)^{n^2/2} z^n$$
$$= \prod_{n=1}^{\infty} \frac{(p^n - q^n)(p^{n-1/2} - q^{n-1/2}z)(p^{n-1/2}z - q^{n-1/2})}{p^{3n-1}z}. \tag{83}$$

The Euler identity follows from the ${}_1\Psi_1$-sum by taking $a = d = 0$, $b = c = 1$, $(p,q) \longrightarrow (1, q^3)$, and $z = q$:

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2} = (q;q)_\infty. \tag{84}$$

8. (p,q)-Special functions

Let us now make some brief observations on the (p,q)-generalizations of the q-special functions. First let us consider an example. It is seen that

$$\left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} = \left[\begin{array}{c} n \\ n-k \end{array} \right]_{p,q} = p^{k(n-k)} \left[\begin{array}{c} n \\ k \end{array} \right]_{q/p} = p^{k(n-k)} \left[\begin{array}{c} n \\ n-k \end{array} \right]_{q/p}. \tag{85}$$

The continuous q-Hermite polynomial is given by

$$H_n(x|q) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{q} e^{i(n-2k)\theta}, \qquad x = \cos\theta. \tag{86}$$

We may define a continuous (p,q)-Hermite polynomial as

$$\mathcal{H}_n(x|p,q) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} e^{i(n-2k)\theta}, \qquad x = \cos\theta. \tag{87}$$

In view of the relation (85) it is found that $\mathcal{H}_n(x|p,q)$ is not just $H_n(x|(q/p))$ with a rescaling of x: *e.g.*, letting $(p,q) \longrightarrow (q^\alpha, q^\beta)$ one would get a two-parameter family of generalized continuous q-Hermite polynomials, say $\left\{ H_n^{(\alpha,\beta)}(x|q) \right\}$ with the usual $H_n(x|q)$ identified as $H_n^{(0,1)}(x|q)$. This is in contrast to the case of ${}_r\Phi_{r-1}$ which can always be identified, as already noted (see (27) and (31)), with an ${}_r\phi_{r-1}$; in this sense, ${}_r\Phi_{r-1}$ may be considered a trivial generalization - examples in this category would be the (p,q)-generalizations of q-Krawtchouk polynomials, q-Meixner polynomials, q-Racah polynomials, q-Askey-Wilson polynomials, q-Jacobi polynomials, q-Hahn polynomials, q-Charlier polynomials, continuous q-ultraspherical polynomials, etc... However, such generalizations are also of interest from the point of view of physical applications: one example of such a situation is the study of the Clebsch-Gordon coefficients of the two-parameter quantum algebra $U_{p,q}(gl(2))$ - a simple relation between the CG-coefficients of $U_{p,q}(gl(2))$ and $U_q(sl(2))$ exists [22] which must be due to the connection between the CG-coefficients of $U_q(sl(2))$ and ${}_3\phi_2$ (see, *e.g.*, [23])). (p,q)-generalizations of gamma and beta functions are straightforward [11]. Besides the continuous q-Hermite polynomials, there are several examples for which the (p,q)-generalization is nontrivial: discrete q-Hermite polynomials, q-Laguerre polynomials, q-Bessel functions $(J_\nu^{(2)}(x;q))$, etc... We hope to return to these topics elsewhere.

9. Conclusion

We have shown that it is profitable to study the (p,q)-hypergeometric series, or the twin-basic hypergeometric series, following naturally from the extension of the q-number $(1-q^n)/(1-q)$ to the twin-basic number $(p^n - q^n)/(p-q)$. In particular, we have studied the (p,q)-analogues of the q-binomial theorem, q-binomial coefficient, Heine transformation for ${}_2\phi_1$, Gauss sum for ${}_2\phi_1$, and the Ramanujan sum for ${}_1\psi_1$. Further, we have made

some brief observations on the (p,q)-generalizations of the q-special functions. In general, we have noted that many of the q-results can be generalized directly to (p,q)-results and once we have the (p,q)-results the q-results can be obtained more easily by mere substitutions for the parameters instead of any limiting process as required in the usual q-theory. We believe that a detailed study of the (p,q)-hypergeometric series, or the twin-basic hypergeometric series, should be very interesting.

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References

[1] R. Chakrabarti and R. Jagannathan, "A (p,q)-oscillator realization of two-parameter quantum algebras", *J. Phys. A: Math. Gen.* **24** (1991) L711-L718.

[2] G. Brodimas, A. Jannussis and R. Mignani, "Two-parameter quantum groups", *Universita di Roma Preprint Nr.* 820 (1991).

[3] M. Arik, E. Demircan, T. Turgut, L. Ekinci and M. Mungan, "Fibonacci oscillators", *Z. Phys. C: Particles and Fields*, **55** (1992) 89-95.

[4] M. Wachs and D. White, "p,q-Stirling numbers and set partition statistics", *J. Combin. Theory A* **56** (1991) 27-46.

[5] J. B. Remmel and M. L. Wachs, "Rook theory, generalized Stirling numbers and (p,q)-analogues", *Electro. J. Combin.* **11** (2004) # R84.

[6] J. Katriel and M. Kibler, "Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers", *J. Phys. A: Math. Gen.* **25** (1992) 2683-2691.

[7] Yu. F. Smirnov and R. F. Wehrhahn, "The Clebsch-Gordan coefficients for the two-parameter quantum algebra $SU_{p,q}(2)$ in the Lowdin-Shapiro approach", *J. Phys. A: Math. Gen.* **25** (1992) 5563-5576.

[8] R. Floreanini, L. Lapointe and L. Vinet, "A note on (p,q)-oscillators and bibasic hypergeometric functions", *J. Phys. A: Math. Gen.* **26** (1993) L611-L614.

[9] R. P. Agarwal and A. Verma, "Generalized basic hypergeometric series with unconnected bases", *Proc. Camb. Phil. Soc.* **63** (1967) 727-734.

[10] R. P. Agarwal and A. Verma, "Generalized basic hypergeometric series with unconnected bases - II", *Quart. J. Math.* **18** (1967) 181-192.

[11] M. Burban and A. U. Klimyk, "P,Q-differentiation, P,Q-integration and P,Q-hypergeometric functions related to quantum groups", *Integral Transforms and Special Functions* **2** (1994) 15-36.

[12] I. M. Gelfand, M. I. Graev and V. S. Retakh, "(r,s)-Hypergeometric functions of one variable", *Russian Acad. Sci. Dokl. Math* **48** (1994) 591-596.

[13] I. M. Gelfand, M. I. Graev and V. S. Retakh, "General gamma functions, exponentials and hypergeometric functions", *Russian Math. Surveys* **53** (1998) 1-55.

[14] R. Jagannathan, "(P, Q)-Special functions", *Proc. the Workshop on Special Functions and Differential Equations* (The Institute of Mathematical Sciences, Chennai, India, January 1997), Eds. K. Srinivasa Rao, R. Jagannathan, G. Vanden Berghe and J. Van der Jeugt (Allied Publishers, New Delhi, 1998) 158-164.

[15] M. Nishizawa, "$U_{r,s}(gl_4)$-symmetry for (r, s)-hypergeometric series", *J. Comp. Appl. Maths.* **160** (2003) 233-239.

[16] V. Sahai and S. Srivastava, "On irreducible p, q-representations of $gl(2)$", *J. Comp. Appl. Maths.* **160** (2003) 271-281.

[17] G. Gasper and M. Rahman, *Basic Hypergeometric Series*,(Cambridge University Press, Cambridge, 1990).

[18] W. N. Bailey, *Generalized Hypergeometric Series*, (Cambridge University Press, Cambridge, 1935).

[19] R. P. Agarwal, *Generalized Hypergeometric Series*, (Asia Publishing House, Bombay, 1963).

[20] L. J. Slater, *Generalized Hypergeometric Series*, (Cambridge University Press, Cambridge, 1966).

[21] T. H. Koornwinder, "q-Special Functions, A Tutorial", *arXiv:math. CA*/9403216.

[22] R. Jagannathan and J. Van der Jeugt, "Finite-dimensional representations of the quantum group $GL_{p,q}(2)$ using the exponential map from $U_{p,q}(gl(2))$", *J. Phys. A: Math. Gen.* **28** (1995) 2819-2831.

[23] K. Srinivasa Rao and V. Rajeswari, *Quantum Theory of Angular Momentum* (Springer-Verlag, 1993).